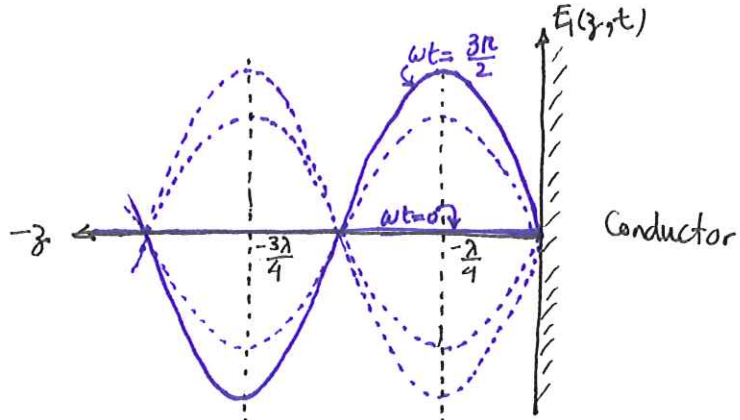
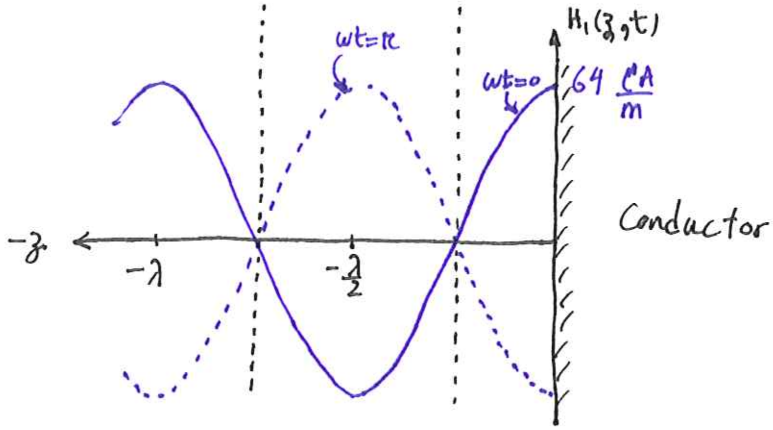
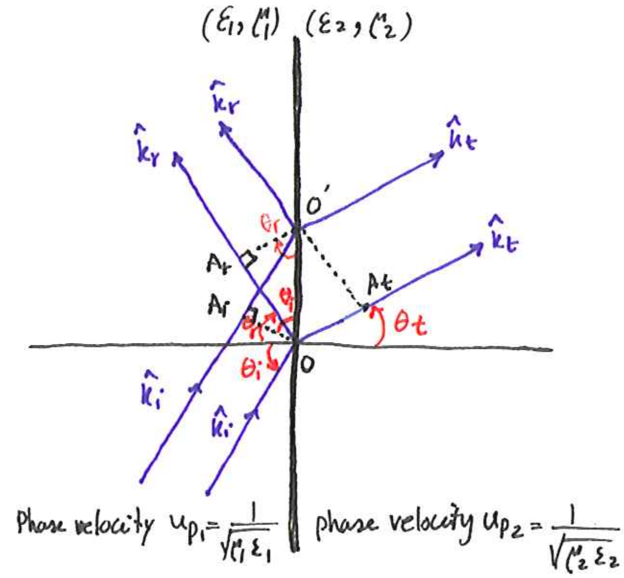




Snell's Laws

We now consider oblique incidence at interface of two materials.

The incident, reflection, and transmission (or refraction) angles $(\theta_i, \theta_r, \theta_t)$ are defined with respect to normal to the boundary.



Time to travel $A_i \longrightarrow O' = O \longrightarrow A_r = O \longrightarrow A_t$

$$\frac{A_iO'}{u_{p_1}} = \frac{OA_r}{u_{p_1}} = \frac{OA_t}{u_{p_2}}$$

$$\frac{OO'\sin\theta_i}{u_{p_1}} = \frac{OO'\sin\theta_r}{u_{p_1}} = \frac{OO'\sin\theta_t}{u_{p_2}} \Longrightarrow$$

$\theta_i = \theta_r$ (Snell's law of reflection)

$$\boxed{\frac{\sin\theta_t}{\sin\theta_i} = \frac{u_{p_2}}{u_{p_1}} = \sqrt{\frac{\mu_1 \varepsilon_1}{\mu_2 \varepsilon_2}}}$$

Index of reflection:

$$\boxed{n = \frac{c}{u_p} = \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} = \sqrt{\mu_r\varepsilon_r}}$$

$$\Rightarrow \frac{\sin\theta_t}{\sin\theta_i} = \frac{n_1}{n_2} = \sqrt{\frac{\mu_{r1}\varepsilon_{r1}}{\mu_{r2}\varepsilon_{r2}}}$$

For nonmagnetic materials $\mu_{r_1} = \mu_{r_2} = 1 \Rightarrow$ $\boxed{\frac{\sin\theta_t}{\sin\theta_i} = \frac{n_1}{n_2} = \sqrt{\frac{\varepsilon_{r1}}{\varepsilon_{r2}}} = \frac{\eta_2}{\eta_1} \quad (\text{for } \mu_1 = \mu_2)}$

note: $\eta = \sqrt{\frac{\mu}{\varepsilon}}$

Usually, denser materials have higher permitivities ε_r. for air $\varepsilon_r = 1$.

Normal incident: $\theta_i = 0 \to \theta_r = 0$ and $\theta_t = 0$ as expected.

Oblique incident: $\theta_t < \theta_i$ if $n_2 > n_1$ and $\theta_t > \theta_i$ if $n_1 > n_2 \Rightarrow$ if a wave enters a denser material, the transmitted wave refracts inwardly toward the normal axis.

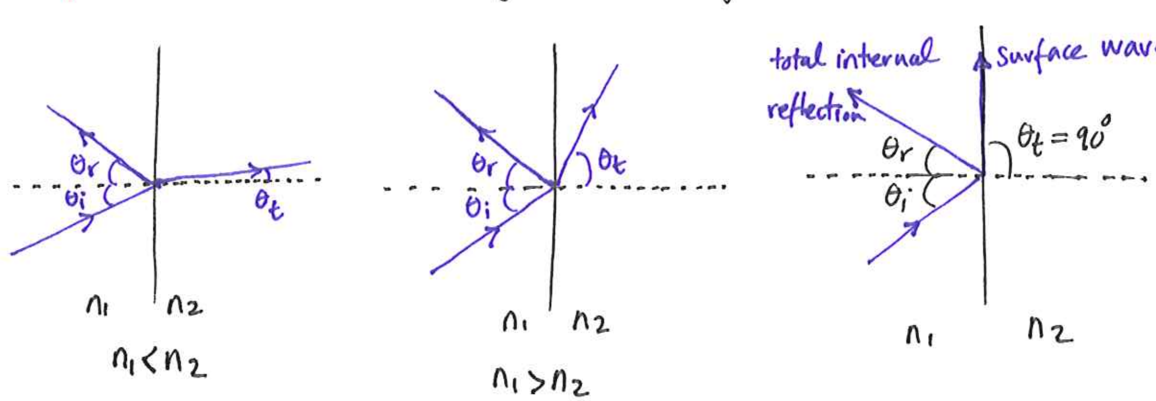
If $\theta_t = 90 \Rightarrow$ no energy enters medium II.

The θ_i corresponding to $\theta_t = 90$ is called critical angle θ_c:

$$\boxed{\sin\theta_c = \frac{n_2}{n_1}\sin\theta_t\Big|_{\theta_t=\frac{\pi}{2}} = \frac{n_2}{n_1} = \sqrt{\frac{\varepsilon_{r2}}{\varepsilon_{r1}}} \quad \text{for } \mu_{r_1} = \mu_{r_2}}$$

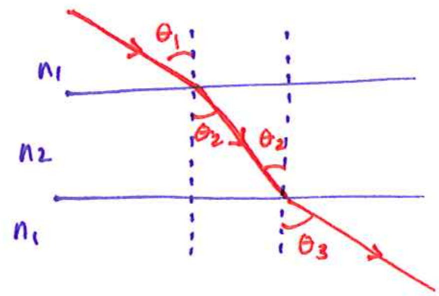
Total internal reflection

If $\theta_i > \theta_c \to$ the incident wave is totally reflected and the reflected wave becomes a nonuniform surface wave that travels along the boundary between the two media.



Example

A dielectric slab is sandwiched between a medium with different index of refraction as shown. Show that the emerging beam is parallel to the incident beam.

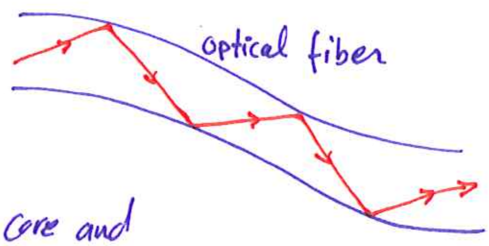


Solution:

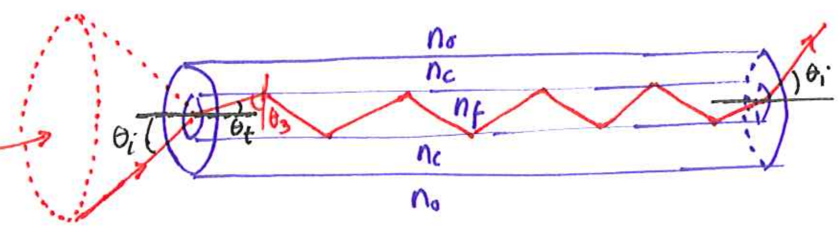
$$\left.\begin{array}{l} n_2 \sin\theta_2 = n_1 \sin\theta_1 \\ n_2 \sin\theta_2 = n_1 \sin\theta_3 \end{array}\right\} n_1 \sin\theta_1 = n_1 \sin\theta_3 \Rightarrow \theta_1 = \theta_3$$

Fiber Optics

Light can be guided through an optical fiber by successive total internal reflections as shown in the picture:



A typical fiber is made of a core and a cladding:



The cladding is to prevent leakage from one fiber to the other when they are packed together.

To make sure that the total internal reflection happens in the fiber:

θ_3 must be greater than θ_c:

$$n_f \sin\theta_c = n_c \sin 90 \Rightarrow \sin\theta_c = \frac{n_c}{n_f}.$$

$$\theta_3 \geq \theta_c \Rightarrow \sin\theta_3 \geq \frac{n_c}{n_f} \qquad \text{Since } \theta_t + \theta_3 = 90^\circ \Rightarrow \cos\theta_t = \sin\theta_3$$

$$\Rightarrow \cos\theta_t \geq \frac{n_c}{n_f} \qquad \text{However: } n_o \sin\theta_i = n_f \sin\theta_t \rightarrow \sin\theta_t = \frac{n_o}{n_f}\sin\theta_i$$

$$\Rightarrow \quad 1 - \left(\frac{n_o}{n_f}\sin\theta_i\right)^2 \geq \frac{n_c^2}{n_f^2}$$

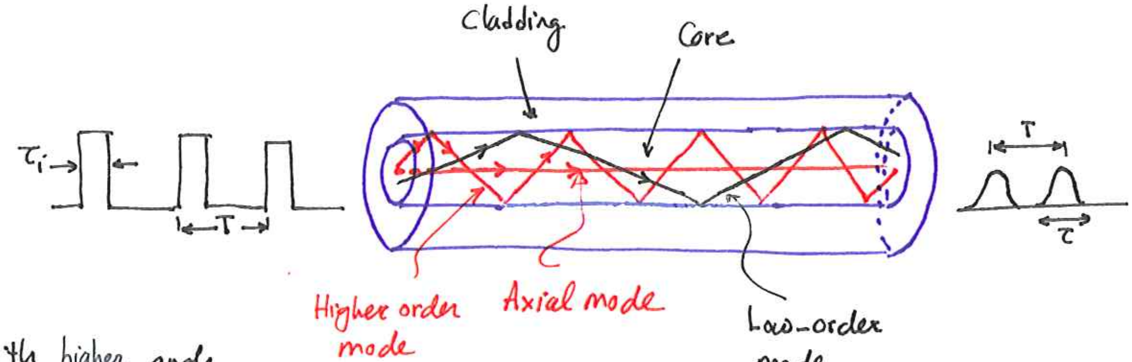
$$\rightarrow \frac{n_0^2}{n_f^2}\sin^2\theta_i \le \frac{n_f^2 - n_c^2}{n_f^2} \rightarrow \sin\theta_i \le \frac{1}{n_0}(n_f^2 - n_c^2)^{1/2}$$

Acceptance angle θ_a:

θ_a is the maximum θ_i for the total internal reflection to happen:

$$\boxed{\sin\theta_a = \frac{1}{n_0}(n_f^2 - n_c^2)^{1/2}}$$

Modal Dispersion



Since the beams with higher angle travel longer distance in the fiber, not all the beams arrive at same time to the end of the fiber. This delay distorts the pulse shape as shown in the picture known as modal dispersion. Each ray path in the fiber is called a mode.

The modal dispersion sets a cap on the maximum pulse frequency that can travel in the fiber. We typically set $T \ge 2\tau$ as shown in the picture.

The spread out width is equal to the time delay between the ray that travels the longest distance in the fiber and the axial mode (normal incident):

$$\begin{cases} \text{Longest travel}: \ \theta_i = \theta_a \Rightarrow \cos\theta_t = \frac{n_c}{n_f} \Rightarrow l_{max} = \frac{l}{\cos\theta_t} = l\frac{n_f}{n_c} \\ \text{shortest travel}: \ \theta_i = 0 \Rightarrow l_{min} = l \end{cases}$$

$$\rightarrow \begin{cases} t_{max} = \frac{l_{max}}{u_p} = \frac{l n_f^2}{c n_c} \\ t_{min} = \frac{l}{u_p} = \frac{l}{c} n_f \end{cases} \rightarrow \tau = \Delta t = t_{max} - t_{min} = \frac{l n_f}{c}\left(\frac{n_f}{n_c} - 1\right)$$

Condition $T \ge 2\tau \Rightarrow f_p = \frac{1}{T} = \frac{1}{2\tau} = \frac{c n_c}{2 l n_f (n_f - n_c)}$ (bits/s)

Maximum frequency

Example Transmission Data Rate on optical Fibers

A 1-km optical fiber (in air) has $n_f = 1.52$ (core) and $n_c = 1.49$ (cladding). Determine

(a) the acceptance angle θ_a

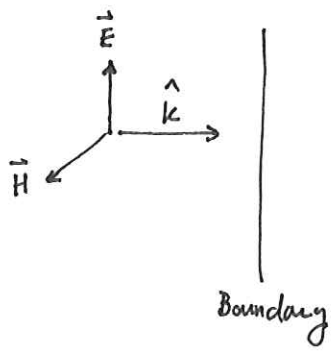
(b) the maximum usable data rate that can be transmitted through the fiber.

Solution: (a) $\sin\theta_a = \frac{1}{n_0}(n_f^2 - n_c^2)^{1/2} = (1.52^2 - 1.49^2)^{1/2} = 0.3 \rightarrow \theta_a = 17.5°$

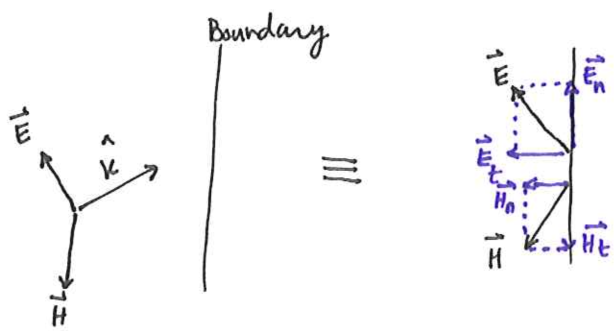
(b) $f_p = \dfrac{c n_c}{2 l n_f (n_f - n_c)} = \dfrac{3\times10^8 \times 1.49}{2\times10^3 \times 1.52(1.52-1.49)} = 4.9\,(\text{Mb/s})$

Wave Reflection and Transmission at Oblique Incident

In normal incident E and H are always tangential to the interface regardless of the polarization:

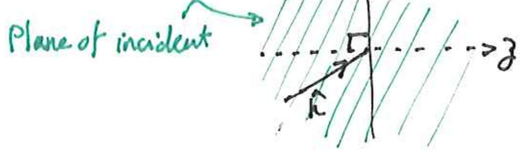


However in oblique incident at an angle $\theta_i \neq 0$, E and H are not tangential to the surface and have both normal and tangential components:

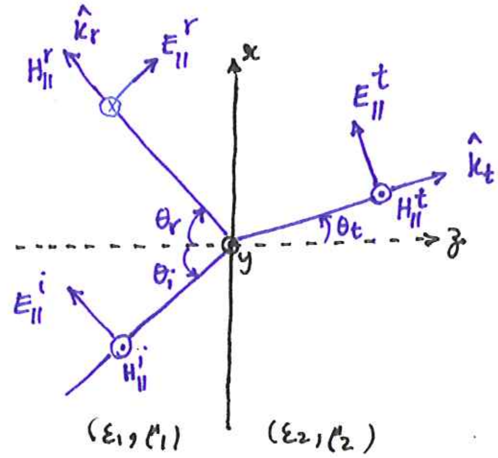


For the electric field, E:
$\vec{E}$ can be described as superposition of two components: (1) parallel to plane of incident i.e. parallel polarization. and perpendicular to the plane of incident called perpendicular polarization.

Plane of incident: The plane containing $\hat{k}$ and the normal to the boundary.



Parallel polarization :

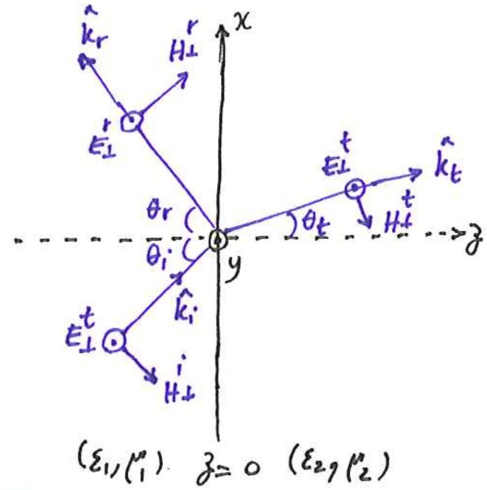


$$\vec{E} = \vec{E}_{\parallel} + \vec{E}_{\perp}$$

$$\vec{H} = H_{\parallel} + \vec{H}_{\perp}$$

$E_{\parallel}$ is parallel to plane of incident $\rightarrow$
Also called transverse magnetic (TM) polarization as $H_{\parallel}$ is normal to the plane.

Perpendicular polarization:



$E_{\perp}$ is perpendicular to the plane of incident $\rightarrow$ also called transverse electric (TE) polarization as $E_{\perp}$ is normal to the plane.

Reflection & Transmission approach for general wave :

For a general wave (E, H) we first decompose the wave into parallel $(E_{\parallel}, H_{\parallel})$ and perpendicular $(E_{\perp}, H_{\perp})$ polarization, and find transmission and reflection components of each (for example $(E_{\perp}^t, H_{\perp}^t)$ & $(E_{\parallel}^t, H_{\parallel}^t)$) and then add them to get the total transmission and reflections.

Perpendicular Polarization

consider the case shown in the picture. $E_{\perp}^i$ is perpendicular to plane (in y direction) and $H_{\perp}^i$ is normal to $E_{\perp}^i$ in y_i direction. $E_{\perp}^i \times H_{\perp}^i$ is in x_i direction which is direction of travel. If $E_{\perp 0}^i$ is the amplitude of $E_{\perp}^i$ at $x_i = 0$, $k_1 = \omega\sqrt{\mu_1 \varepsilon_1}$ is the wavenumber, and $\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}$ the intrinsic impedance, we have:

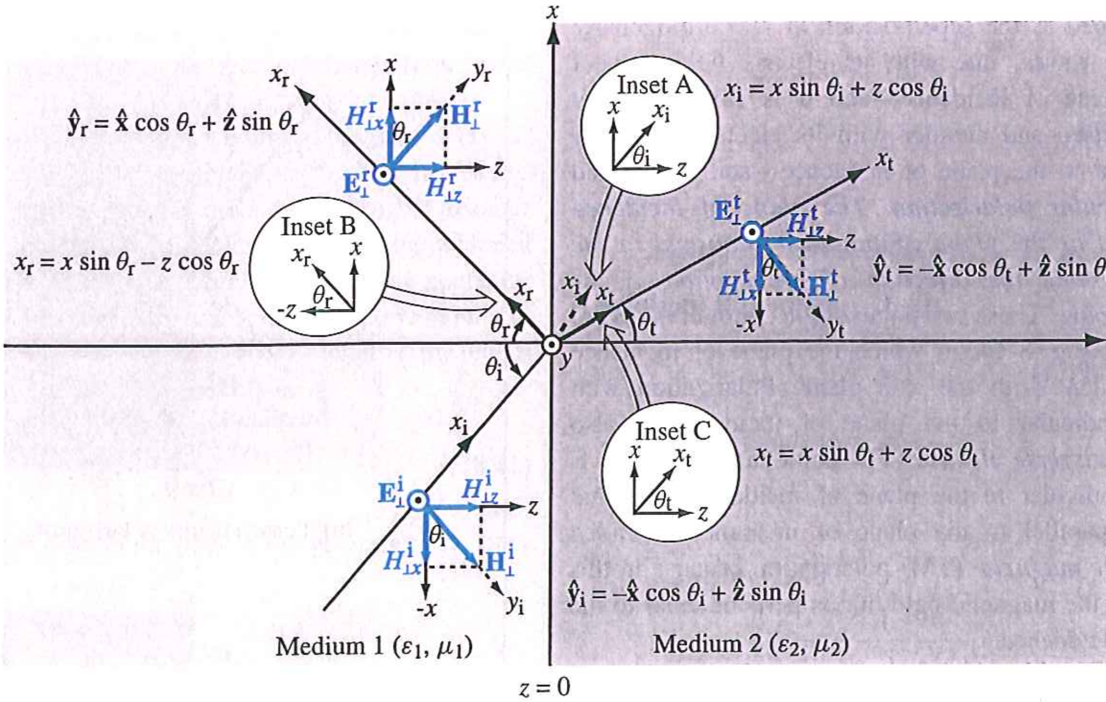


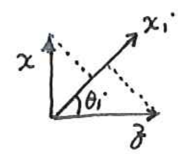
Figure 8-15: Perpendicularly polarized plane wave incident at an angle θ_i upon a planar boundary.

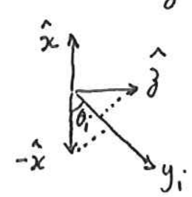
Incident wave:

$$\begin{cases} \vec{\tilde{E}}^i_\perp = \hat{y}\, E^i_{\perp 0}\, e^{-jk_1 x_i} \\ \vec{\tilde{H}}^i_\perp = \hat{y}_i\, \dfrac{E^i_{\perp 0}}{\eta_1}\, e^{-jk_1 x_i} \end{cases}$$

x_i and y_i can be decomposed in x, y, z plane as:

$$\begin{cases} x_i = x\sin\theta_i + z\cos\theta_i \\ \hat{y}_i = -\hat{x}\cos\theta_i + \hat{z}\sin\theta_i \end{cases}$$





Substitute in $\vec{\tilde{E}}^i_\perp$ and $\vec{\tilde{H}}^i_\perp$ =>

$$\begin{cases} \vec{\tilde{E}}^i_\perp = \hat{y}\, E^i_{\perp 0}\, e^{-jk_1(x\sin\theta_i + z\cos\theta_i)} \\ \vec{\tilde{H}}^i_\perp = (-\hat{x}\cos\theta_i + \hat{z}\sin\theta_i)\, \dfrac{E^i_{\perp 0}}{\eta}\, e^{-jk_1(x\sin\theta_i + z\cos\theta_i)} \end{cases}$$

Reflected wave:

We can similarly write down the expressions for $\tilde{E}^r_\perp$, $\tilde{H}^r_\perp$ and $\tilde{E}^t_\perp$, $\tilde{H}^t_\perp$:

$$\begin{cases} \vec{\tilde{E}}^r_\perp = \hat{y}\, E^r_{\perp 0}\, e^{-jk_1 x_r} = \hat{y}\, E^r_{\perp 0}\, e^{-jk_1(x\sin\theta_r - z\cos\theta_r)} \\ \vec{\tilde{H}}^r_\perp = \hat{y}_r\, \dfrac{E^r_{\perp 0}}{\eta_1}\, e^{-jk_1 x_r} = (\hat{x}\cos\theta_r + \hat{z}\sin\theta_r)\, \dfrac{E^r_{\perp 0}}{\eta_1}\, e^{-jk_1(x\sin\theta_r - z\cos\theta_r)} \end{cases}$$

Transmitted wave:

$$\vec{\tilde{E}}^t_{\perp} = \hat{y}\, E^t_{\perp o}\, e^{-jk_2 x_t} = \hat{y}\, E^t_{\perp o}\, e^{-jk_2(x \sin\theta_t + z\cos\theta_t)}$$

$$\vec{\tilde{H}}^t_{\perp} = \hat{y}_t\, \frac{E^t_{\perp o}}{\eta_2}\, e^{-jk_2 x_t} = (-\hat{x}\cos\theta_t + \hat{z}\sin\theta_t)\, \frac{E^t_{\perp o}}{\eta_2}\, e^{-jk_2(x\sin\theta_t + z\cos\theta_t)}$$

we have four unknowns to find: $E^r_{\perp o}$, $E^t_{\perp o}$, θ_r, θ_t

Although θ_r and θ_t are related to θ_i by Snell's law, we keep the unknown to derive here.

To find the unknowns we must apply the boundary conditions:

1) Tangential components of $\vec{\tilde{E}}$ and $\vec{\tilde{H}}$ (as there is no current at interface) are equal at $z=0$. So for $\vec{\tilde{E}}$ we have:

$$\vec{\tilde{E}}_{1\,tangant}\Big|_{z=0} = \vec{\tilde{E}}_{2\,tangant}\Big|_{z=0}$$

$$\tilde{E}^i_{\perp y} + \tilde{E}^r_{\perp y}\Big|_{z=0} = \tilde{E}^t_{\perp y}\Big|_{z=0}$$

$$E^i_{\perp o}\, e^{-jk_1 x \sin\theta_i} + E^r_{\perp o}\, e^{-jk_1 x\sin\theta_r} = E^t_{\perp o}\, e^{-jk_2 x \sin\theta_r} \qquad \text{I}$$

For $\vec{\tilde{H}}$ we have:

$$\left(\tilde{H}^i_{\perp x} + \tilde{H}^r_{\perp x}\right)\Big|_{z=0} = \tilde{H}^t_{\perp x}\Big|_{z=0}$$

$$-\frac{E^i_{\perp o}}{\eta_1}\cos\theta_i\, e^{-jk_1 x\sin\theta_i} + \frac{E^r_{\perp o}}{\eta_1}\cos\theta_r\, e^{-jk_1 x\sin\theta_r} = -\frac{E^t_{\perp o}}{\eta_2}\cos\theta_t\, e^{-jk_2 x\sin\theta_t} \qquad \text{II}$$

To satisfy I and II at all values of x, the three exponents must be equal:

$$k_1\sin\theta_i = k_1\sin\theta_r = k_2\sin\theta_t \quad \rightarrow \text{phase-matching Condition}$$

Snell's law of reflection: $\theta_i = \theta_r$

$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{k_1}{k_2} = \frac{\omega\sqrt{\mu_1\varepsilon_1}}{\omega\sqrt{\mu_2\varepsilon_2}} = \frac{n_1}{n_2}$$ snell's law of transmission

$\Rightarrow$ I and II reduce to: $\begin{cases} E_{\perp 0}^{i} + E_{\perp 0}^{r} = E_{\perp 0}^{t} \\ \dfrac{\cos\theta_i}{\eta_1}\left(-E_{\perp 0}^{i} + E_{\perp 0}^{r}\right) = -\dfrac{\cos\theta_t}{\eta_2} E_{\perp 0}^{t} \end{cases} \Rightarrow$

$$\boxed{\begin{aligned} \Gamma_\perp &= \frac{E_{\perp 0}^{r}}{E_{\perp 0}^{i}} = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t} \\ \tau_\perp &= \frac{E_{\perp 0}^{t}}{E_{\perp 0}^{i}} = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i + \eta_1\cos\theta_t} \end{aligned}}$$

Fresnel reflection and transmission coefficients for perpendicular polarization.

we can show that $\boxed{\tau_\perp = 1 + \Gamma_\perp}$ as expected.

If medium 2 is a perfect conductor $(\eta_2 = 0) \rightarrow \Gamma_\perp = -1$ and $\tau_\perp = 0$, which means the incident wave is totally reflected by the conducting medium.

For non-magnetic materials $\mu_1 = \mu_2 = \mu_0$ and we can simplify $\Gamma_\perp$ to:

$$\Gamma_\perp = \frac{\cos\theta_i - \sqrt{\frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_i}}{\cos\theta_i + \sqrt{\frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_i}} \quad (\mu_1 = \mu_2) \qquad \text{note that } \frac{\varepsilon_2}{\varepsilon_1} = \left(\frac{n_2}{n_1}\right)^2$$

$$= \frac{\cos\theta_i - \sqrt{\frac{n_2^2}{n_1^2} - \sin^2\theta_i}}{\cos\theta_i + \sqrt{\frac{n_2^2}{n_1^2} - \sin^2\theta_i}}$$

Example

Consider electric field $E^i = \hat{y}\, 100 \cos(\omega t - \pi x - 1.73\pi z)$ is incident in air upon a plane soil at $z = 0$. Soil is lossless dielectric with $\varepsilon_r = 4$.

(a) determine k_1, k_2, θ_i

(b) Find expression for $\vec{E}$ in air and soil

(c) Find the average power carried by the wave traveling in the soil.

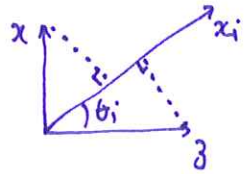
Solution:

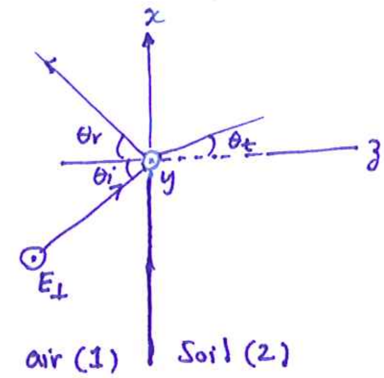
(a) $\tilde{\vec{E}}^i = \hat{y}\, 100\, e^{-j\pi x}\, e^{-j1.73\pi z} = \hat{y}\, 100\, e^{-jk_1 x_i}$ $\left(\frac{V}{m}\right)$ x_i is the axis of incident.

$\Rightarrow \quad k_1 x_i = \pi x + 1.73\pi z$

We had: $\quad k_1 x_i = k_1 x \sin\theta_i + k_1 z \cos\theta_i$

$$\begin{cases} k_1 \sin\theta_i = \pi \\ k_1 \cos\theta_i = 1.73\pi \end{cases} \rightarrow \quad k_1 = \sqrt{\pi^2 + (1.73\pi)^2} = 2\pi, \quad \theta_i = \tan^{-1}\frac{\pi}{1.73\pi} = 30^\circ$$





In medium 1 i.e. air: $\lambda_1 = \frac{2\pi}{k_1} = \frac{2\pi}{2\pi} = 1\,m$

In medium 2 i.e. soil: $\lambda_2 = \frac{\lambda_1}{\sqrt{\varepsilon_r}} = \frac{1}{\sqrt{4}} = 0.5\,m$

$\rightarrow k_2 = \frac{2\pi}{\lambda_2} = \frac{2\pi}{0.5} = 4\pi$

(b) $k_1 \sin\theta_i = k_2 \sin\theta_t \rightarrow \sin\theta_t = \frac{k_1}{k_2}\sin\theta_i = \frac{2\pi}{4\pi}\sin 30 = 0.25 \rightarrow \theta_t = 14.5^\circ$

$$\begin{cases} \Gamma_\perp = \dfrac{\cos\theta_i - \sqrt{\varepsilon_2/\varepsilon_1 - \sin^2\theta_i}}{\cos\theta_i + \sqrt{\frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_i}} = -0.38 \\ \tau_\perp = 1 + \Gamma_\perp = 0.62 \end{cases}$$

In air:

$\rightarrow \tilde{\vec{E}}_\perp^1 = \tilde{\vec{E}}_\perp^i + \tilde{\vec{E}}_\perp^r = \hat{y} E_{\perp 0}^i e^{-jk_1(x\sin\theta_i + z\cos\theta_i)} + \hat{y}\,\Gamma E_{\perp 0}^i e^{-jk_1(x\sin\theta_i - z\cos\theta_i)}$

$= \hat{y}\, 100\, e^{-j(\pi x + 1.73\pi z)} - \hat{y}\, 38\, e^{-j(\pi x - 1.73\pi z)}$

$\vec{E}_\perp^1(x,z,t) = \hat{y}\left[100\cos(\omega t - \pi x - 1.73\pi z) - 38\cos(\omega t - \pi x + 1.73\pi z)\right]$

In Soil:

$\tilde{\vec{E}}_\perp^t = \hat{y}\,\tau E_{\perp 0}^i e^{-jk_2(x\sin\theta_t + z\cos\theta_t)} = \hat{y}\, 62\, e^{-j(\pi x + 3.87\pi z)}$

$\vec{E}_\perp^t(x,y,t) = \hat{y}\, 62\cos(\omega t - \pi x - 3.87\pi z)$

(c) $S_{av}^t = \frac{|E_{\perp 0}^t|^2}{2\eta_2} = \frac{62^2}{2 \times 60\pi} = 10.2 \left(\frac{W}{m^2}\right) \Leftarrow \eta_2 = \frac{\eta_0}{\sqrt{\varepsilon_r}} \simeq \frac{120\pi}{\sqrt{4}} = 60\pi$